

# Microcanonical Entropy of Isolated Horizon and the Barbero-Immirzi parameter

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The *microcanonical* entropy of a generic Isolated Horizon(IH) is shown to be completely determined in terms of the independent hairs which are required for a complete characterization of an IH. For the Bekenstein-Hawking area law to be valid, the Barbero-Immirzi parameter( $\gamma$ ) needs to be a function of the independent hairs.  $\gamma$  lies within the range  $0.159 < \gamma < 0.225$  and is *not* a free parameter. Remarkably, there is no  $\gamma$ -ambiguity in the quantum theory of spacetimes *admitting physical boundaries*.

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In quantum geometry of black hole spacetime[1, 2], the *intersections* of the bulk spin network edges with the classical Isolated Horizon(IH)[3–7], called *punctures*, depict a quantum IH(QIH). The boundary states of a quantum black hole are that of the SU(2) Chern-Simons(CS) theory coupled to the *punctures* of the QIH. Each of the punctures is associated with a half-integral SU(2) spin with a maximum value of  $k/2$ ,  $k$  being the level of the CS theory. The counting of the SU(2) microstates in [8] has led to the successful calculation of *microcanonical* entropy of a black hole, considering all spins to be  $1/2$ , which results in the semiclassical Bekenstein-Hawking area law(BHAL)[9] for a particular *fit* of the Barbero-Immirzi(BI) parameter ( $\gamma$ ), alongside a quantum log correction with an universal co-efficient  $-3/2$ [10]. Later on, this result has been generalized considering spin 1 [11] and also for general spin distribution [12–15], which only alters the value of the BI parameter. The strength of this QIH framework lies in the fact that it provides a unique self contained platform for the direct application of statistical mechanics to explore black hole thermodynamics. Generally, the only semi-classical input which is needed to fix the  $\gamma$ -ambiguity of the underlying theory, is the BHAL.

In recent literature [16], it has been shown that, apart from the BHAL, some additional inputs are required to derive the *microcanonical* entropy of a black hole (e.g. assumption of a specific energy spectrum). The result, of course, introduces a new term  $N\sigma$  in the expression for the entropy of black holes (having the specific energy spectrum), where  $N$  is the total number of punctures on the IH and  $\sigma$  is a function of  $\gamma$ . It has been claimed that  $\gamma$  is a free parameter.

In this letter, we show that the *microcanonical* entropy of a *generic* black hole (irrespective of its energy spectrum), along with the new term( $N\sigma$ ), is

completely derivable within the QIH framework using the standard statistical mechanical method of finding most probable distribution[18] followed in [16]. Any extra input, other than the usual BHAL, is redundant. For the BHAL to be valid,  $\gamma$  comes out to be a *function* of the ratio of the quantum hairs of the IH. Graphical plot of the function, along with some information theoretic arguments, reveal that  $\gamma$  *is not a free parameter* but has a finite range of allowed values. As a remarkable consequence, there is no  $\gamma$ -ambiguity in the quantum theory of spacetimes admitting *physical boundaries*.

A generic QIH is described by an SU(2) spin configuration given by the set  $\{s_j\}$ , where  $s_j$  is the number of punctures with spin value  $j$ . The number of microstates for a particular spin configuration  $\{s_j\}$  in the SU(2) approach[8, 13, 14] is given by

$$\Omega[\{s_j\}] = \frac{2}{k+2} \frac{N!}{\prod_j s_j!} \times \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{s_j} \quad (1)$$

The combinatorial factor arises from the choice of *distinguishable* punctures [13, 14].

Now, an IH being a *non-expanding* horizon [3–7], is characterized by its classical area  $A_{cl}$ . Here, we are interested only in the microstates (QIHs) with area eigenvalues  $A = A_{cl} \pm \mathcal{O}(\ell_p^2)$ . Since, for macroscopic black holes  $A_{cl} \gg \ell_p^2$ , for all practical purposes, we can consider  $A \approx A_{cl}$  neglecting the  $\mathcal{O}(\ell_p^2)$  fluctuations and keeping only the contribution from the most probable distribution. Hence, while defining the microcanonical ensemble, fixing  $A_{cl}$  is equivalent to fixing  $A$ . This argument follows from standard statistical mechanics[18]. The subtlety arises due to the presence of the unknown BI parameter in the area spectrum of QIH. Truly speaking, we can fix only  $A/\gamma$ . Hence,  $A/8\pi\gamma\ell_p^2 \approx A_{cl}/8\pi\gamma\ell_p^2 = \mathcal{A}$  is the appropriate choice of an independent parameter of the quantum system, a quantum hair. The choice is justified by the fact that  $\mathcal{A} = k/2$  is nothing but the

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level of the CS theory (upto a factor of  $1/2$ ) [1, 19], which by itself is an independent parameter of the theory. *This is in contrast to all the earlier works in literature [1, 2, 10, 13–16, 20–26] where microcanonical entropy of black hole is calculated, in LQG framework, for fixed  $A_{cl}$  which is equivalent to fixing  $\mathcal{A}$  only if  $\gamma$  is a priori considered to be a fixed, unknown number.* Exception occurs only in [27], where  $k$  has been considered as a more fundamental parameter rather than  $A_{cl}$  and  $\gamma$  is not a fixed number but a function of  $k$  only.

Now, having identified the relevant quantum hairs, we define a *microcanonical ensemble* of QIHs by fixing the values of  $N$  and  $\mathcal{A}$ . Hence, the spin configuration  $\{s_j\}$  must obey the following constraints

$$\mathcal{C}_1 : \sum_{j=1/2}^{k/2} s_j = N \quad (2a)$$

$$\mathcal{C}_2 : \sum_{j=1/2}^{k/2} s_j \sqrt{j(j+1)} = \mathcal{A} \quad (2b)$$

where  $\mathcal{A} = A/8\pi\gamma\ell_p^2 \approx A_{cl}/8\pi\gamma\ell_p^2 = k/2$  [1, 19],  $A$  being the area spectrum of the QIH in LQG [28]. Variation of  $\log \Omega[\{s_j\}]$  with respect to  $s_j$ , subject to the constraints  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , yields the most probable distribution which maximizes the entropy of the IH. The variational equation is given by

$$\delta \log \Omega[\{s_j\}] - \sigma \delta N - \lambda \delta \mathcal{A} = 0 \quad (3)$$

where  $\delta$  represents variation with respect to  $s_j$ ,  $\sigma$  and  $\lambda$  are the Lagrange multipliers for  $\mathcal{C}_1$  and  $\mathcal{C}_2$  respectively. This yields the most probable distribution given by

$$\bar{s}_j = N M_j(k) e^{-\lambda \sqrt{j(j+1)} - \sigma} \quad (4)$$

where

$$\begin{aligned} M_j(k) &= \prod_{a=1}^{k+1} \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{\frac{f_a(k)}{f(k)}} \\ f(k) &= \sum_{a=1}^{k+1} f_a(k) \\ &= \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{\bar{s}_j} \end{aligned}$$

Eq.(1) can be rewritten as

$$\Omega[\{s_j\}] = \frac{2}{k+2} \frac{N!}{\prod_j s_j!} f(k) \quad (5)$$

Hence, from eq.(5), for the dominant configuration  $\{\bar{s}_j\}$ , whose distribution is given by eq.(4), we have

$$\log \Omega[\{\bar{s}_j\}] = \log \frac{2f(k)}{k+2} + \log N! - \sum_j \log(\bar{s}_j!)$$

Since we are investigating macroscopic black holes, we consider large number of punctures for each spin value. Henceforth, we take the limit  $\bar{s}_j \rightarrow \infty$  for all the calculations. So, one can calculate the *microcanonical entropy* ( $S_{MC}$ ) in the appropriate limit and show that

$$\begin{aligned} S_{MC} &= \lim_{\bar{s}_j \rightarrow \infty} \log \Omega[\{\bar{s}_j\}] \\ &= \lambda \mathcal{A} + \sigma N + \log \frac{2f(k)/(k+2)}{\prod_j \{M_j(k)\}^{\bar{s}_j}} \end{aligned}$$

using Stirling's approximation,  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . In the limit  $k \rightarrow \infty$  for macroscopic black holes, the logarithmic term appearing on the right hand side leads to the usual  $-3/2 \log A$  quantum correction. Right now we will ignore this logarithmic term as it does not affect our calculations. Henceforth, we will consider only

$$S_{MC} = \lambda \mathcal{A} + \sigma N \quad (6)$$

Now, to get the BHAL, we need to *fit* the BI parameter by choosing  $\gamma = \lambda/2\pi$ . As a result the expression for the *microcanonical entropy* comes out to be

$$S_{MC} = \frac{A}{4\ell_p^2} + N\sigma \quad (7)$$

Now, there is a crucial point to be understood about  $S_{MC}$  in eq.(7) which in terms of  $\mathcal{A}$  and  $N$  is given by  $S_{MC} = 2\pi\gamma\mathcal{A} + N\sigma$ . Given the values of the hairs  $\mathcal{A}$  and  $N$ , we still do not know the exact *microcanonical entropy* of the IH due to the presence of  $\gamma$  and  $\sigma$  whose values are unknown. But, a careful observation reveals that  $\gamma$  and  $\sigma$  can be expressed in terms of  $\mathcal{A}$  and  $N$ . One can show by using eq.(4) in  $\mathcal{C}_1, \mathcal{C}_2$  and putting  $\lambda = 2\pi\gamma$ , that the two unknowns  $\gamma$  and  $\sigma$  satisfy the following two equations

$$e^\sigma = \sum_j M_j(k) e^{-2\pi\gamma\sqrt{j(j+1)}} \quad (8)$$

$$\frac{\mathcal{A}}{N} = \frac{\sum_j M_j(k) \sqrt{j(j+1)} e^{-2\pi\gamma\sqrt{j(j+1)}}}{e^\sigma} \quad (9)$$

Having two equations and two unknowns we can solve the equations as follows. Eliminating  $\sigma$  from the eq.(8) and eq.(9) one obtains

$$\frac{\mathcal{A}}{N} = \frac{\sum_j M_j(k) \sqrt{j(j+1)} e^{-2\pi\gamma\sqrt{j(j+1)}}}{\sum_j M_j(k) e^{-2\pi\gamma\sqrt{j(j+1)}}} \quad (10)$$

For simplicity we consider the following limit :  $\lim_{k \rightarrow \infty} M_j(k) = (2j+1)$ , as a consequence of which the eq.(10) reduces to

$$\frac{\mathcal{A}}{N} = \frac{\sum_j (2j+1) \sqrt{j(j+1)} e^{-\lambda \sqrt{j(j+1)}}}{\sum_j (2j+1) e^{-\lambda \sqrt{j(j+1)}}} \quad (11)$$

Since  $j$  runs from  $1/2$  to  $k/2$ , for macroscopic black holes  $k$  can be taken to be large enough so as to treat  $j$  as a continuous variable. Thus, replacing the summation by integration and taking the appropriate limits of integration for  $j$ , eq.(11) can be written as

$$\frac{\mathcal{A}}{N} = \frac{\int_0^\infty (2j+1) \sqrt{j(j+1)} e^{-2\pi\gamma\sqrt{j(j+1)}} dj}{\int_0^\infty (2j+1) e^{-2\pi\gamma\sqrt{j(j+1)}} dj - 1} \quad (12)$$

Performing the integrations and doing a bit of algebra eq.(12) is reduced to the following simple form

$$2\pi^3 a \gamma^3 - \pi a \gamma + 1 = 0 \quad (13)$$

where  $a = \mathcal{A}/N$ , the ratio of the quantum hairs. Since, for a given  $\mathcal{A}$  and  $N$  we have a particular value of  $a$ , we need to solve the cubic equation for  $\gamma$  in terms of the parameter  $a$ . The only real solution of  $\gamma$  comes out to be as follows

$$\gamma = \frac{6^{1/3} a^2 + [-9a^2 + 3^{1/2} a^2 (27 - 2a^2)^{1/2}]^{2/3}}{6^{2/3} \pi a [-9a^2 + 3^{1/2} a^2 (27 - 2a^2)^{1/2}]^{1/3}} \quad (14)$$

To find the allowed values of the BI parameter ( $\gamma$ ), we plot  $\gamma$  vs  $a$  in FIG.(1), FIG.(2) and FIG.(3) for suitable ranges of the parameters.

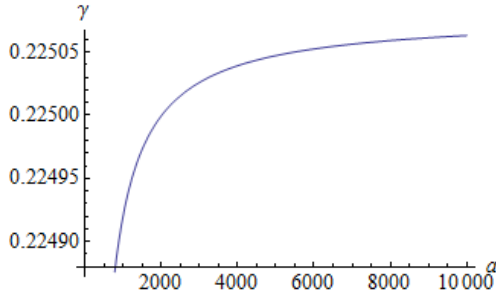


FIG. 1: The graph shows the variation of  $\gamma$  with  $a$  for high values of  $a$ . It is seen clearly that the value of  $\gamma$  saturates at approximately 0.225 as  $a$  increases indefinitely. Hence,  $\gamma_{max} \approx 0.225$ .

Now, applying the approximation for large  $k$  i.e.  $\lim_{k \rightarrow \infty} M_j(k) = (2j+1)$  and replacing the sum over  $j$  by integration, keeping track of the appropriate limits in eq.(8), one finds after evaluation of the integral that  $e^\sigma = 2/(2\pi\gamma)^2 - 1$ . Hence,  $\sigma$  in terms of  $\mathcal{A}$  and  $N$  is given by

$$\sigma(\mathcal{A}, N) = \log \left\{ \frac{1}{2\pi^2 \gamma^2(\mathcal{A}, N)} - 1 \right\} \quad (15)$$

Note that for  $\sigma$  to be real, we must have  $\gamma < 1/\sqrt{2\pi} \approx 0.225$ . This is consistent with the plot

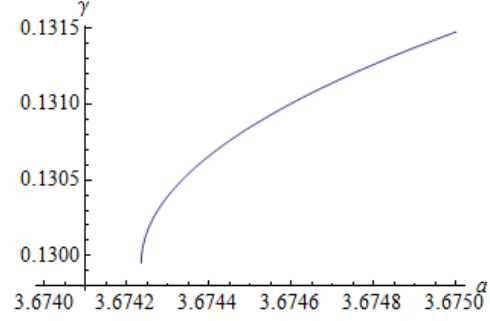


FIG. 2: The graph shows the variation of  $\gamma$  with  $a$  for low values of  $a$ . It is clearly observable that  $\gamma$  has real positive solutions for  $a > 3.674$ . The lowest real positive value of  $\gamma$  estimated from the above graph is approximately 0.130.

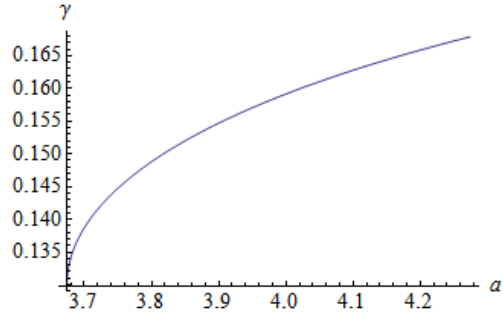


FIG. 3: The above plot of  $\gamma$  vs  $a$  reveals that, for  $\gamma_{min} \approx 0.159$  we have  $a \approx 4.000$ .

of  $a$  vs  $\gamma$  in FIG.(1) where  $\gamma$  saturates at 0.225 as  $a$  increases. Also,  $\gamma$  itself must have a real positive value. Studying lower portion of the  $\gamma$  vs  $a$  plot at a magnified scale in FIG.(2), one can find that this happens for  $a > 3.674$  with the lowest real positive value of  $\gamma$  being 0.130. But, this is not the actual lower bound on  $\gamma$ . By information theoretic arguments, one can estimate the actual lower bound on  $\gamma$  as follows. Imposing constraints on a system implies availability of more information about that system. Since, entropy is a measure of uncertainty i.e. unavailability of information about a system, imposition of more constraints will result in decrement of the entropy [29]. Black hole being a physical system, the above argument is also true for the calculation of black hole entropy. In fact, the availability of extra quantum information is the exact reason for the *negative* logarithmic correction to semi-classical BHAL as has been properly explained in [10]. Similar implications of the constraints follow for the black hole entropy calculation shown here. If one calculates the entropy for fixed  $\mathcal{A}$  only, the result will be the usual BHAL (ignoring log correction), with  $\gamma$  being a function of  $\mathcal{A}$  only [27]. Now, if one

fixes  $N$  alongside  $\mathcal{A}$  i.e. a further constraint is imposed on the system, there must be a *negative* term in addition to the BHAL (alongside log correction ignored here). Thus the new term appearing in the expression of the entropy must be negative definite i.e.  $N\sigma(\gamma) < 0$ . Since,  $N$  being the total number of punctures on the IH can only be positive. Hence, there is no other choice than  $\sigma$  to be *negative definite* i.e.  $\sigma < 0$ . Now, as far as the allowed values of  $\gamma$  are concerned, the bound on  $\sigma$  modifies the lower bound of  $\gamma$ . The explicit form of  $\sigma$  in terms  $\gamma$  is given by  $\sigma = \log(1/2\pi^2\gamma^2 - 1)$ . Hence,  $\sigma < 0$  implies that  $(1/2\pi^2\gamma^2 - 1) < 1$  i.e.  $\gamma > 1/2\pi \approx 0.159$ . Thus, the appropriate range of the allowed values of  $\gamma$  is given by  $1/2\pi < \gamma < 1/\sqrt{2}\pi$  or up to three places of decimal the approximate range is  $0.159 < \gamma < 0.225$ . From the plot in FIG. (3), the corresponding range of  $a$  is given by  $4.000 < a < \infty$ .

Hence, considering the BHAL to be valid, we have expressed  $\gamma$  and  $\sigma$  in terms of the preassigned values of  $\mathcal{A}$  and  $N$  given by eq.(14) and eq.(15), respectively, which allow us to write  $S_{MC}$  completely in terms of the *independent* hairs  $\mathcal{A}$  and  $N$  as

$$S_{MC}(\mathcal{A}, N) = 2\pi\gamma(\mathcal{A}, N)\mathcal{A} + N\sigma(\mathcal{A}, N) \quad (16)$$

Thus, once a *microcanonical* ensemble of QIHs is defined by preassigning the values of  $\mathcal{A}$  and  $N$ , the *microcanonical* entropy of the IH is completely known. This is consistent with the identification of  $\mathcal{A}$  and  $N$  as the *true independent quantum hairs characterizing an IH*.

Here, one may be worried about the preciseness of the calculation of the range of  $\gamma$  because we deliberately do an approximation by replacing summation by integration for simplicity. A numerical calculation may yield a more precise result which will only be a minor correction since the approximation is good enough for macroscopic black holes. But the point we want to show is that  $\gamma$  is *not a free parameter and has a specific range of allowed values*. One can note that the allowed range of values reported here is not something ad hoc and comparable to all the existing values of the BI parameter reported earlier in literature [1, 2, 10–16, 20–27, 30] as far as the order of magnitude is concerned.

Just for clarification, let us recollect that one can recast eq.(16) in the familiar form, manifesting the BHAL, as

$$S_{MC} = \frac{A}{4\ell_p^2} + N\sigma$$

where all the quantities are known in terms of the preassigned values of  $\mathcal{A}$  and  $N$ . This, radically dif-

fers from what has been obtained in [16], where  $\gamma$  remains a *free parameter* in contrast to our result.

It is important to note that, since we are interested in retrieving the BHAL in eq.(7), we have applied the fit  $\lambda = 2\pi\gamma$  and then solved for  $\gamma$  and  $\sigma$  from the available equations. However, if one does not mind in recovering the BHAL in eq.(7), he/she does not need to choose  $\lambda = 2\pi\gamma$ . In that case  $\gamma$  is retained as an independent unknown free parameter. The solutions available are that of  $\lambda$  and  $\sigma$  in terms of  $\mathcal{A}$  and  $N$ . Thus  $S_{MC}$  becomes a function of  $\mathcal{A}$ ,  $N$  and  $\gamma$  i.e. even if one preassigns the values of  $\mathcal{A}$  and  $N$ , there remains a  $\gamma$  ambiguity in the *microcanonical* entropy and also BHAL is not retrieved. Since, we are interested only in the BHAL, this particular scenario is not an issue of interest here.

Now, let us shed some light on a very different aspect of the our findings. Though the introduction of  $\mathcal{A}$  and  $N$  as true *independent* quantum hairs unravels the complete structure of the *microcanonical* entropy of an IH in terms of two *independent* hairs, there is an even deeper significance of these two *independent* quantum hairs of the IH. The BI parameter  $\gamma$  appears in the theory of canonical quantization of gravitational degrees of freedom for the full spacetime including bulk and boundary[28]. Interestingly, according to the results obtained here,  $\gamma$  has a well defined value *only for spacetimes with physical boundaries* whose quantum description incorporates the definition of the boundary hairs  $\mathcal{A}$  and  $N$ . Since  $\gamma$  is expressed in terms of  $\mathcal{A}$  and  $N$ , physical boundaries of spacetimes must play a crucial role in the quantum theory of the associated bulk. As a consequence, physical boundaries are absolutely essential for spacetimes to have unambiguous quantum theories. Thus, for spacetimes *admitting physical boundaries*, the boundary and bulk quantum theories are knotted by the BI parameter  $\gamma(\mathcal{A}, N)$ , a function of boundary quantum hairs, which will keep track of the consistency between the two theories. But, for spacetimes *without* physical boundaries, there will always be the usual  $\gamma$  - ambiguity in the quantum theory as before. This leads to the only conclusion that *physical boundaries of spacetimes are absolutely indispensable for having an unambiguous quantum theory of spacetime*, provided we consider the BHAL to be valid.

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